Math 1B – Chapter 8.5 – 8.9 Test – Fall '08 Name______________________

Show your work for credit. Write all responses on separate paper.

1. Find by differentiating the function $f(x)=\sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$. Remember to show your work. Use the result to approximate $\sqrt{1.2}$

2. Find the radius of convergence and the interval of convergence of the series

   a. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^{2n}}$

   b. $\displaystyle\sum_{n=0}^{\infty}\frac{n^5(x-1)^{2n}}{5^n}$.

3. Find the Taylor series for $\ln(5+x^2)$ using term-by-term differentiation or integration of the appropriate geometric series. Give enough terms to make the pattern clear.

4. Find a Taylor series for the Fresnel function, $\int_0^x \sin(t^2)dt$. What is the interval of convergence?

5. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for $y=\dfrac{x}{\sin x}$.

6. Use the binomial theorem to approximate $\sqrt[6]{730}=\sqrt[6]{1+3^6}=3\left(1+\dfrac{1}{3^6}\right)^{1/6}$ by computing the fourth degree Taylor approximating polynomial, $T_4(x)$ for $f(x)=(1+x)^{1/6}$ about $a = 0$ and evaluating $\sqrt[6]{760}\approx 3T_4\left(\dfrac{1}{3^6}\right)$.

7. Consider $f(x)=\cos(x)$.

   a. Approximate f by a Taylor polynomial with degree 4 around $a = \pi/3$.

   b. Use Taylor's Inequality to extimate the accuracy of the approximation $f(x)\approx T_4(x)$ when $0\le x\le\dfrac{2\pi}{3}$

   c. Use the Alternating Series Estimation Therorem to estimate the range of values for which the approximation $f(x)\approx T_4(x)$ is accurate to within $|\text{error}| < 0.0001$

Math 1B – Chapter 8.5 – 8.9 Test Solutions – Fall '08

1. Find by differentiating the function $f(x)=\sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$. Remember to show your work. Use the result to approximate $\sqrt{1.2}$

   SOLN: $f'(x)=\frac{1}{2}(1+x)^{-1/2}$; $f''(x)=-\frac{1}{4}(1+x)^{-3/2}$; $f'''(x)=\frac{3}{8}(1+x)^{-5/2}$ so

   $c_0=f(0)=1;\ c_1=f'(0)=\frac{1}{2};\ c_2=\frac{f''(0)}{2}=-\frac{1}{8};\ c_3=\frac{f'''(0)}{6}=\frac{1}{16}$

   So that $f(x)=\sqrt{1+x}\approx 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$ and $\sqrt{1.2}\approx T_3(0.2)=1+\frac{1}{10}-\frac{1}{200}+\frac{1}{2000}=1.0955$

2. Find the radius of convergence and the interval of convergence of the series

   a. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^{2n}}$ SOLN: $\lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{2(n+1)}}\frac{3^{2n}}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{9}\right|=\left|\frac{x}{9}\right|<1\Leftrightarrow |x|<9$. At $x = 9$, $\sum_{n=0}^{\infty}\frac{(-1)^n 9^n}{3^{2n}}=\sum_{n=0}^{\infty}(-1)^n$ does not converge. Also, at $x = -9$, $\sum_{n=0}^{\infty}\frac{(-1)^n(-9)^n}{3^{2n}}=\sum_{n=0}^{\infty}1$ is divergent, so the interval of convergence is $(-9, 9)$.

   b. $\sum_{n=0}^{\infty}\frac{n^5(x-1)^{2n}}{5^n}$. SOLN: $\lim_{n\to\infty}\left|\frac{(n+1)^5(x-1)^{2(n+1)}}{5^{(n+1)}}\frac{5^n}{n^5(x-1)^{2n}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^5}{n^5}\frac{(x-1)^{2(n+1)}}{(x-1)^{2n}}\frac{5^n}{5^{(n+1)}}\right|$ $=\left|\frac{(x-1)^2}{5}\right|<1\Leftrightarrow 1-\sqrt{5}<x<1+\sqrt{5}$. Now, at both endpts, $\sum_{n=0}^{\infty}\frac{n^5(x-1)^{2n}}{5^n}=\sum_{n=0}^{\infty}n^5$ is divergent, so the interval of convergence does not include either endpoint.

3. Find the Taylor series for $\ln(5+x^2)$ using term-by-term differentiation or integration of the appropriate geometric series. Give enough terms to make the pattern clear.

   SOLN: $\frac{d}{dx}\ln(5+x^2)=\frac{2x}{5}\frac{1}{1+x^2/5}=\frac{2x}{5}\sum_{n=0}^{\infty}\left(-\frac{x^2}{5}\right)^n=2\sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}x^{2n+1}$. Now we integrate,

   $$\ln(5+x^2)=c+\int\frac{d}{dx}\ln(5+x^2)=c+2\int\sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}x^{2n+1}=\ln 5+\sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}(n+1)}x^{2n+2}$$

   So the first few terms are $\ln 5+\frac{x^2}{5}-\frac{x^4}{50}+\frac{x^6}{375}-\frac{x^8}{2500}+\frac{x^{10}}{15625}-\cdots$

4. Find a Taylor series for the Fresnel function, $\int_0^x \sin\left(t^2\right)dt$. What is the interval of convergence?

SOLN: $\frac{d}{dx}\int_0^x \sin\left(t^2\right)dt = \sin x^2 = \sum_{n=0}^{\infty}\frac{\left(x^2\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{4n+2}}{(2n+1)!}$ so that

$$\int_0^x \sin\left(t^2\right)dt = \int\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!} = c+\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} = \frac{x^3}{3}-\frac{x^7}{42}+\frac{x^{11}}{1320}-\frac{x^{15}}{75600}+\cdots$$

5. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for $y=\frac{x}{\sin x}$.

SOLN: The terms are $1+\frac{x^2}{6}+\frac{7x^4}{360}$

6. Use the binomial theorem to approximate $\sqrt[6]{730}=\sqrt[6]{1+3^6}=3\left(1+\frac{1}{3^6}\right)^{1/6}$ by computing the fourth degree Taylor approximating polynomial, $T_4(x)$ for $f(x)=(1+x)^{1/6}$ about $a=0$ and evaluating $\sqrt[6]{760}\approx 3T_4\left(\frac{1}{3^6}\right)$.

SOLN: $f(x)=(1+x)^{1/6}=\sum_{n=0}^{\infty}\binom{1/6}{n}x^n \approx T_4(x)=1+\frac{x}{6}-\frac{5}{72}x^2+\frac{55}{1296}x^3-\frac{935}{31104}x^4$

$$\sqrt[6]{730}=\sqrt[6]{1+3^6}=3\left(1+\frac{1}{3^6}\right)^{1/6}\approx 3\left(1+\frac{1}{6\cdot 3^6}-\frac{5}{72\bullet 3^{12}}+\frac{55}{1296\cdot 3^{18}}-\frac{935}{31104\cdot 3^{24}}\right)$$

Thus, $$=3\left(1+\frac{1}{4374}-\frac{5}{38263752}+\frac{55}{502096953744}-\frac{935}{8784688302705024}\right)$$

$$\approx 3\begin{pmatrix}1+0.00022862368541381\\ -0.00000013067197383\\ +0.00000000010954060\\ -0.00000000000010643\end{pmatrix}\approx 3.0006854793686224$$

Whereas, $\sqrt[6]{730}\approx 3.0006854793686227$ is a better approximation.

7. Consider $f(x)=\cos(x)$.
   a. Approximate *f* by a Taylor polynomial with degree 4 around $a = \pi/3$.
   SOLN:
$$\cos(x) \approx \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) - \frac{\cos\left(\frac{\pi}{3}\right)}{2}\left(x-\frac{\pi}{3}\right)^2 + \frac{\sin\left(\frac{\pi}{3}\right)}{6}\left(x-\frac{\pi}{3}\right)^3 + \frac{\cos\left(\frac{\pi}{3}\right)}{24}\left(x-\frac{\pi}{3}\right)^4$$
$$\approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\left(x-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x-\frac{\pi}{3}\right)^4$$
   b. Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_4(x)$ when $0 \le x \le \frac{2\pi}{3}$.
   SOLN: Since the interval includes π/2, the maximum value of the absolute value of the fifth derivative is $M = 1$. Thus $\left|R_4(x)\right| \le \frac{1\cdot(\pi/3)^5}{5!} \approx 0.0104945$

   c. Use the Alternating Series Estimation Theorem to estimate the range of values for which the approximation $f(x) \approx T_4(x)$ is accurate to within |error| < 0.0001
   SOLN: This is a bit awkward for two reasons: (1) to the right of *a*, the series does not alternate term by term, but every pair of terms is alternating, and (2) to the left of *a* the series alternates pairs of terms but we're in the middle of a pair instead of after a pair.

   Let's ignore problem (2) for now and concentrate on (1). What we'll do is take the sum of the absolute values of the two negative terms that follow the last two positive terms as a bound on the error:
$$\left|c_5\left(x-\frac{\pi}{3}\right)^5 + c_6\left(x-\frac{\pi}{3}\right)^6\right| = \left|-\frac{\sqrt{3}}{240}\left(x-\frac{\pi}{3}\right)^5 - \frac{1}{1440}\left(x-\frac{\pi}{3}\right)^6\right| \le \frac{\sqrt{3}}{240}\left(\frac{\pi}{3}\right)^5 + \frac{1}{1440}\left(\frac{\pi}{3}\right)^6 \approx 0.010004$$
   a somewhat tighter bound on the error. To be sure, the true error is
$$\left|\cos\left(\frac{2\pi}{3}\right) - T_4\left(\frac{2\pi}{3}\right)\right| \approx 0.009752890$$
   Grouping the terms like so:
$$\cos(x) \approx \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\left(\frac{\pi}{3}-x\right) - \frac{1}{4}\left(\frac{\pi}{3}-x\right)^2\right) - \left(\frac{\sqrt{3}}{12}\left(\frac{\pi}{3}-x\right)^3 - \frac{1}{48}\left(x-\frac{\pi}{3}\right)^4\right) + \left(\frac{\sqrt{3}}{240}\left(\frac{\pi}{3}-x\right)^5 - \frac{1}{1440}\left(x-\frac{\pi}{3}\right)^6\right)$$
$$\approx 0.5 + 0.632744 - 0.140701 + 0.008173$$
   we see this as an alternating series of differences and the next difference is only 0.008173 – smaller than the difference on the right. But this is ok since $\left|\cos(0) - T_4(0)\right| \approx 0.007956681837$